

Genericity in multivariate cryptography

(or how to appease Reviewer B by removing heuristics)

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Multivariate signature schemes: a template

- **Public key:** a system of polynomial equations of degree two → Hard to solve
- **Signature:** a solution of the public system of polynomial equations
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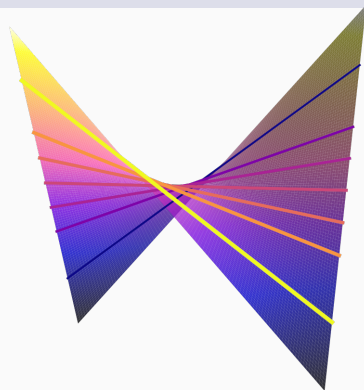
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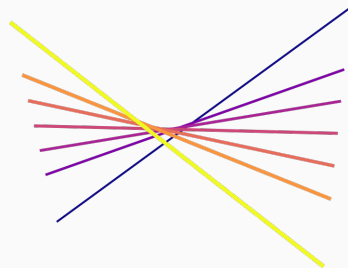
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$$(1, 0, 0) + \mathbb{R}(0, 1, 1).$$



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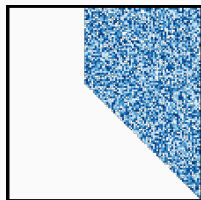
Original (U)OV formulation

Private key: polynomials $f_1, \dots, f_m \in \mathbb{F}_q[X_1, \dots, X_n]$ linear in $X_1, \dots, X_o$, $A \in GL_n(\mathbb{F}_q)$.

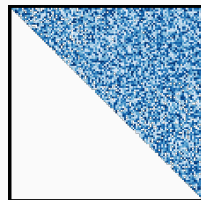
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Private polynomial

$$F_1 \in (\mathbb{F}_{257})^{n \times n}$$



$$A \in GL_n(\mathbb{F}_{257})$$



Public polynomial

$$P_1 \in (\mathbb{F}_{257})^{n \times n}$$

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Equivalent formulation: there exists a linear subspace $\mathcal{O}$ of dim. o such that

$$p_1(\mathcal{O}) = \dots = p_m(\mathcal{O}) = 0.$$

For UOV, require that $o \geq m$.

Genericity in the Zariski topology

Zariski topology on $\mathbb{K}^n$

A property is **Zariski-generic** if it holds on a non-empty Zariski-open set.

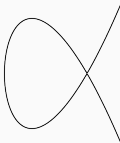
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Example: generic smoothness

The curve of equation $Y^2 - X^3 + 3X - 2 = 0$ is **singular**.
Are all curves of equation $Y^2 - X^3 - aX - b = 0$ singular?



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Figure 1: Curves in $\mathbb{R}^2$ defined by $Y^2 - X^3 - aX - 2$ for $a \in [-3.4, -2.6]$.

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Methodology

Consider a, b as **variables** and study the equation $\text{Jac}_E(a, b) = (0, 0)$.

$$\text{Jac}_E(a, b) = (0, 0) \in \mathbb{K}(X, Y)[a, b] \implies \delta(a, b) = 4a^3 + 27b^2 = 0.$$

$$\text{Jac}_{\mathcal{F}} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial X_1} & \cdots & \frac{\partial f_m}{\partial X_n} \end{bmatrix}$$

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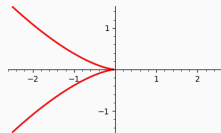
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Therefore, if $\delta(a, b)$ is non-zero, $E_{a,b}$ is **smooth**.



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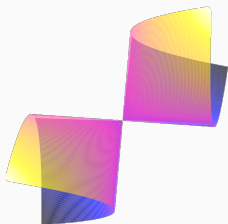
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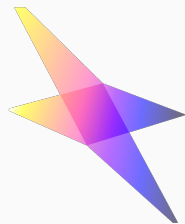
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The theorem is projective: we work in $\mathbb{P}^2(\mathbb{R})$.



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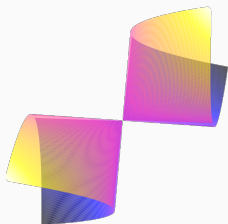
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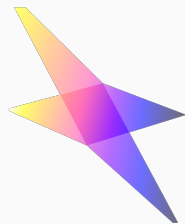
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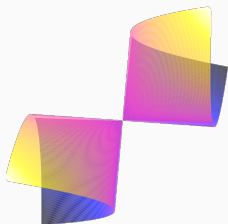
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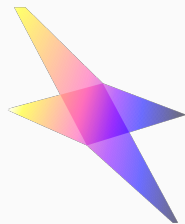
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Thus, a **generic** quadric hypersurface in $\mathbb{R}^3$ contains **lines** but does not contain **planes**.



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$$1 \leq k \leq m, 1 \leq i \leq j \leq o, \quad P_k(X_i, X_j) = 0.$$
- The dimension of $\mathbb{G}_o(\mathbb{P}^n(\mathbb{K}))$ is $(o + 1)(n - o)$, ie the number of coefficients in an echelon basis of an $(o + 1)$ -dimensional linear subspace of $\mathbb{K}^{n+1}$.

Singularities of generic UOV keys.

[P. 2025]

For a **generic** UOV key $\mathcal{P} \in \mathcal{U}$, we have $\text{Sing}(V(\mathcal{I})) \subset \mathcal{O}$.

This result unlocks subsequent genericity results.

Contributions and Outline of the talk

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A family of examples in odd characteristic.

This work

The previous result holds in any finite field of odd characteristic.

Address a limit of [P. 2025] which worked only over large fields.

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Support analysis of the direct attack and of some key recovery attacks. In particular, geometric attacks are well-defined.

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...more exotic assumptions on MinRank also are.

This work

For a **generic** UOV key $\mathcal{P} \in \mathcal{M}$, the determinantal ideals related to the Jacobian evaluated on $\mathcal{O}$ have the expected dimension.

Support the analysis of Rectangular MinRank attacks. No false positives in the Kipnis-Shamir attack.

For **generic** UOV keys, all singularities lie in $\mathcal{O}$.

Structured equations yield a structured Jacobian. [P. 2025]

The Jacobian of $\mathcal{F}$ is structured when $\mathbf{x} \in \mathcal{O}$:

$$\text{Jac}_{\mathcal{F}}(\mathbf{x}) = \begin{matrix} & \begin{matrix} 1 & \dots & o & o+1 & \dots & n \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ m \end{matrix} & \left[\begin{array}{cc|c} \text{red } 0 & & J_2 \end{array} \right] \end{matrix}$$

- There are singularities in $\mathcal{O}$.
- Are there singular points outside of $\mathcal{O}$, that may hinder attacks or enable distinguishers?

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Generic smoothness of a singular variety.

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It suffices to give examples in smaller fields to extend the result by proving $\mathcal{U} \neq \emptyset$.

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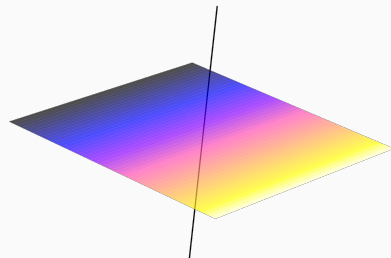
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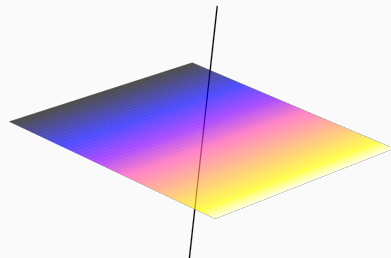
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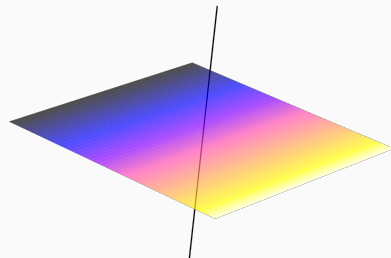
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Supporting heuristics in the algebraic analysis of attacks.

Direct attack against UOV

Given $\mathbf{t} = \mathcal{H}(M) \in \mathbb{F}_q^m$, find $\mathbf{x} \in \mathbb{F}_q^n$ such that

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To solve, reduce to a zero-dimensional system.

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- **Irregular** UOV keys (ie $n - m < o$) are not regular sequences (early MAYO parameters, QR-UOV/SNOVA big field instances, VOX, ...).

Determinantal varieties of UOV and application to MinRank attacks.

Dimension of determinantal varieties

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Generic determinantal ideals of UOV and dimension of the tangent spaces **This work**

Consider (n, m, o, q) with $n \geq 2o$ and $n - m > o$. There exists a Zariski-open set of UOV keys $\mathcal{M}_r$ such that for all $\mathcal{F} \in \mathcal{M}_r$, the ideal $I(\mathcal{O}) \cap \text{Minors}_{r+1}(\text{Jac}_{\mathcal{F}})$ has the expected dimension.

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- Jacobian determinantal ideals of UOV have the expected dimension. [SFINU 26]
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Experimental results.

Method

Compute Gröbner bases of ideals defined by UOV keys sampled uniformly at random in finite fields of various size and characteristic, with 1000 samples in each case.

n, m, o	r	$\mathcal{D}_{16}$	$\mathcal{D}_{31}$	$\mathcal{D}_{251}$	$\mathcal{D}_{1021}$	$\mathcal{D}_{4093}$
10, 4, 4	1	93.3%	96.5%	99.8%	100%	99.9%
10, 5, 5	1	99.9%	99.9%	100%	100%	100%
	2	93.1%	97.6%	99.4%	99.7%	100%
12, 5, 5	1	99.5%	100%	100%	100%	100%
12, 7, 5	1	100%	99.9%	100%	100%	100%
	2	94.4%	97.6%	99.6%	99.8%	100%
12, 6, 6	1	100%	100%	100%	100%	100%
	2	99.4%	100%	100%	100%	100%
14, 6, 6	1	100%	100%	100%	100%	100%
14, 8, 6	1	100%	99.6%	100%	100%	100%

Figure 2: Proportion of UOV keys in the Zariski open sets $\mathcal{M}_{m-r}$.

Experimental results.

Method

Compute Gröbner bases of ideals defined by UOV keys sampled uniformly at random in finite fields of various size and characteristic, with 1000 samples in each case.

When $r = 1$, we obtain a result on both $\mathcal{U}$ and $\mathcal{M}_{m-r}$. $\mathcal{U}$ can also be studied by running Kipnis-Shamir attacks.

n, m, o	r	$\mathcal{D}_{16}$	$\mathcal{D}_{31}$	$\mathcal{D}_{251}$	$\mathcal{D}_{1021}$	$\mathcal{D}_{4093}$
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10, 5, 5	1	99.9%	99.9%	100%	100%	100%
	2	93.1%	97.6%	99.4%	99.7%	100%
12, 5, 5	1	99.5%	100%	100%	100%	100%
12, 7, 5	1	100%	99.9%	100%	100%	100%
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Remark on UOV public keys forming regular sequences.

Widely reproduced experiments in the literature (see e.g. [Faugère, Perret 09]).

Unsurprisingly, we are also able to reproduce these results (with 100% success, even in $\mathbb{F}_2$).

Method: sample UOV keys uniformly and compute the Hilbert series of the sequence.

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Preliminary version in Chap. 5 of my thesis: <https://theses.hal.science/tel-05605746>.

Experiments and code on GitHub soon, ask for code by email!